

DOES EVERY CONTRACTIVE ANALYTIC FUNCTION IN A POLYDISK HAVE A DISSIPATIVE n -DIMENSIONAL SCATTERING REALIZATION?

MICHAEL T. JURY

ABSTRACT. No.

The title question was posed by D. Kalyuzhnyi-Verbovetskyi [1, Problem 1.3]. Let $L(\mathcal{H}, \mathcal{K})$ denote the set of all bounded linear operators between a pair of Hilbert spaces \mathcal{H}, \mathcal{K} , and let \mathbb{D}^n and \mathbb{T}^n denote the open unit polydisk, and the unit n -torus, respectively.

Definition 1. An *dissipative nD scattering system* is a tuple

$$(1) \quad \alpha = (n; \mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}; \mathcal{X}, \mathcal{U}, \mathcal{Y})$$

where:

- i) $n \geq 1$ is an integer;
- ii) $\mathcal{X}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces;
- iii) $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are n -tuples of operators (so $\mathbf{A} = (A_1, \dots, A_n)$, etc.) with

$$(2) \quad A_k \in L(\mathcal{X}, \mathcal{X}), \quad B_k \in L(\mathcal{U}, \mathcal{X}), \quad C_k \in L(\mathcal{X}, \mathcal{Y}), \quad D_k \in L(\mathcal{U}, \mathcal{Y});$$

- iv) The operator $\zeta \mathbf{G} \in L(\mathcal{X} \oplus \mathcal{U}, \mathcal{X} \oplus \mathcal{Y})$ is contractive for all ζ in the unit n -torus \mathbb{T}^n , where

$$(3) \quad \zeta \mathbf{G} := \sum_{k=1}^n \zeta_k G_k$$

and the G_k are the 2×2 block operators

$$(4) \quad G_k = \begin{pmatrix} A_k & B_k \\ C_k & D_k \end{pmatrix}$$

Given such a system, its *transfer function* is the $L(\mathcal{U}, \mathcal{Y})$ -valued function

$$(5) \quad \theta_\alpha(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B}.$$

defined for all $z \in \mathbb{D}^n$. It is shown in [3] that the transfer function θ_α is a *contractive operator function*; that is, it is analytic in the unit polydisk \mathbb{D}^n and satisfies

$$(6) \quad \|\theta_\alpha(z)\|_{L(\mathcal{U}, \mathcal{Y})} \leq 1$$

for all $z \in \mathbb{D}^n$. The question is then whether every contractive operator function in \mathbb{D}^n , vanishing at the origin, is such a transfer function. The answer is known to be “yes” when $n = 1$ or 2 , and in fact a stronger result is true: \mathbf{G} can be chosen so that $\zeta \mathbf{G}$ is unitary (that is, the scattering system is *conservative*). It was also known that when $n = 3$, there exist contractive operator functions which do not have conservative realizations; this is due to the failure of von Neumann’s inequality in three variables. (See [1, 3] for a discussion.) In this note we show the answer is still “no” in the dissipative case when $n = 3$, and give an explicit counterexample (in the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$).

We first show that any polynomial with a dissipative realization must satisfy a restricted form of von Neumann’s inequality. Let \mathcal{T} denote the set of all n -tuples of commuting operators $\mathbf{T} =$

Date: January 17, 2012.

Research partially supported by NSF grant DMS 1101134.

(T_1, \dots, T_n) on Hilbert space satisfying the following condition: whenever $\mathbf{X} = (X_1, \dots, X_n)$ is an n -tuple of operators satisfying

$$(7) \quad \left\| \sum_{k=1}^n z_k X_k \right\| \leq 1$$

for all $z = (z_1, \dots, z_n) \in \mathbb{D}^n$, then

$$(8) \quad \left\| \sum_{k=1}^n T_k \otimes X_k \right\|_{L(\mathcal{H} \otimes \mathcal{K})} \leq 1$$

where the T_k act on \mathcal{H} and the X_k act on \mathcal{K} .

It is easy to see that the \mathbf{T} satisfying this condition must be commuting contractions, but when $n \geq 3$ it is known that not every n -tuple of commuting contractions belongs to \mathcal{T} .

Theorem 2. *If p is a polynomial which can be realized as the transfer function of a dissipative nD scattering system, then*

$$(9) \quad \|p(\mathbf{T})\| \leq 1$$

for all $\mathbf{T} \in \mathcal{T}$.

We will say such p satisfy the *restricted von Neumann inequality*.

Proof of Theorem 2. Suppose p is a polynomial vanishing at 0 and $p = \theta_\alpha$ for some α as in Definition 1; we work only in the scalar case $\mathcal{U} = \mathcal{Y} = \mathbb{C}$. First note that since analytic functions in the polydisk satisfy a maximum principle relative to \mathbb{T}^n , the dissipativity condition (iv) implies

$$(10) \quad \|z\mathbf{G}\| := \left\| \sum_{k=1}^n z_k G_k \right\| \leq 1$$

for all $z \in \mathbb{D}^n$. Then by definition, if $\mathbf{T} \in \mathcal{T}$, we have

$$(11) \quad \left\| \sum_{k=1}^n T_k \otimes G_k \right\| \leq 1.$$

Next, we recall the classical fact that if

$$(12) \quad F = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$$

is a block operator and $\|F\| < 1$, then the linear fractional operator

$$(13) \quad Z + Y(I - W)^{-1}X$$

is contractive. Now apply this to the block operator

$$(14) \quad r\mathbf{T} \cdot \mathbf{G} = \begin{pmatrix} r\mathbf{T} \cdot \mathbf{A} & r\mathbf{T} \cdot \mathbf{B} \\ r\mathbf{T} \cdot \mathbf{C} & r\mathbf{T} \cdot \mathbf{D} \end{pmatrix}$$

where $\mathbf{T} \cdot \mathbf{A} := \sum_{k=1}^n T_k \otimes A_k$, etc., and $r < 1$. We conclude that if $\mathbf{T} \in \mathcal{T}$, then the linear fractional operator

$$(15) \quad r\mathbf{T} \cdot \mathbf{D} + r\mathbf{T} \cdot \mathbf{C}(I_{\mathcal{H} \otimes \mathcal{X}} - r\mathbf{T} \cdot \mathbf{A})^{-1}r\mathbf{T} \cdot \mathbf{B}$$

is contractive for all $r < 1$. But it is straightforward to check that, since p is assumed to be given by the transfer function realization

$$(16) \quad p(z) = z\mathbf{D} + z\mathbf{C}(I_{\mathcal{X}} - z\mathbf{A})^{-1}z\mathbf{B},$$

the expression (15) is equal to $p(r\mathbf{T})$ (This can be done by expanding (16) in a power series, substituting $r\mathbf{T}$ for z , and comparing coefficients with the expansion of (15) in powers of rT_1, \dots, rT_n .) But then $\|p(r\mathbf{T})\| \leq 1$ for all $\mathbf{T} \in \mathcal{T}$ and $r < 1$, which suffices to establish the theorem. \square

It follows that any contractive polynomial which fails the restricted von Neumann inequality will fail to have a dissipative realization. In fact, the counterexample to the classical von Neumann inequality produced by Kaijser and Varopoulos is, it turns out, also a counterexample to the restricted inequality, as we now show. The computations are taken from a closely related example considered in [2].

Let e_1, \dots, e_5 denote the standard basis of \mathbb{C}^5 . Consider the unit vectors

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{3}}(-e_2 + e_3 + e_4) \\ v_2 &= \frac{1}{\sqrt{3}}(e_2 - e_3 + e_4) \\ v_3 &= \frac{1}{\sqrt{3}}(e_2 + e_3 - e_4) \end{aligned}$$

The Kaijser-Varopoulos contractions are the commuting 5×5 matrices T_1, T_2, T_3 defined by

$$T_j = e_{j+1} \otimes e_1 + e_5 \otimes v_j$$

If p is the polynomial

$$(17) \quad p(z_1, z_2, z_3) = \frac{1}{5}(z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3)$$

then it is known that $\sup_{\zeta \in \mathbb{T}^3} |p(\zeta)| = 1$ but

$$(18) \quad \|p(\mathbf{T})\| = \frac{3\sqrt{3}}{5} > 1,$$

so p fails the classical von Neumann inequality [5]. To show that this p fails the restricted von Neumann inequality, we show that already this \mathbf{T} belongs to \mathcal{T} ; that is, if X_1, X_2, X_3 are operators which satisfy

$$(19) \quad \|z_1X_1 + z_2X_2 + z_3X_3\| \leq 1$$

for all $z \in \mathbb{D}^n$, then $\|\sum_{k=1}^3 T_k \otimes X_k\| \leq 1$. To see this, we compute and find

$$(20) \quad T_1 \otimes X_1 + T_2 \otimes X_2 + T_3 \otimes X_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ X_1 & 0 & 0 & 0 & 0 \\ X_2 & 0 & 0 & 0 & 0 \\ X_3 & 0 & 0 & 0 & 0 \\ 0 & Y_1 & Y_2 & Y_3 & 0 \end{pmatrix}$$

where

$$\begin{aligned} Y_1 &= \frac{1}{\sqrt{3}}(-X_1 + X_2 + X_3) \\ Y_2 &= \frac{1}{\sqrt{3}}(X_1 - X_2 + X_3) \\ Y_3 &= \frac{1}{\sqrt{3}}(X_1 + X_2 - X_3) \end{aligned}$$

The norm of the matrix (20) is equal to the maximum of the norms of the first column and the last row. By (19), we have $\|\pm X_1 \pm X_2 \pm X_3\| \leq 1$ for all choices of signs, so the last row of (20) has norm at most 1. To say that the first column has norm at most 1 amounts to saying that

$$(21) \quad I - \sum_{k=1}^n X_k^* X_k \geq 0.$$

This may be seen by averaging: by (19), the matrix valued function

$$I - \sum_{i,j=1}^n \zeta_i \bar{\zeta}_j X_j^* X_i$$

is positive semidefinite on \mathbb{T}^n . Integrating against normalized Lebesgue measure on \mathbb{T}^n gives (21).

There is a general principle that transfer function realizations should be equivalent to von Neumann-type inequalities. Some recent, general results in this direction may be found in [2, 4].

REFERENCES

- [1] Vincent D. Blondel and Alexandre Megretski, editors. *Unsolved problems in mathematical systems and control theory*. Princeton University Press, Princeton, NJ, 2004.
- [2] Michael T. Jury. Universal commutative operator algebras and transfer function realizations of polynomials. <http://arxiv.org/abs/1009.6219>.
- [3] Dmitriy S. Kalyuzhniy. Multiparametric dissipative linear stationary dynamical scattering systems: discrete case. *J. Operator Theory*, 43(2):427–460, 2000.
- [4] Meghna Mittal and Vern I. Paulsen. Operator algebras of functions. *J. Funct. Anal.*, 258(9):3195–3225, 2010.
- [5] N. Th. Varopoulos. On an inequality of von Neumann and an application of the metric theory of tensor products to operators theory. *J. Functional Analysis*, 16:83–100, 1974.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, BOX 118105, GAINESVILLE, FL 32611-8105, USA
E-mail address: `mjury@ufl.edu`